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An anisotropic cosmological model with Bianchi type III universe in f(G) gravity

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Abstract

The paper is devoted to exploring the exact solution of the modified Einstein's field in the setting of spatially homogeneous and anisotropic Bianchi type III spacetime in modified $f(G)$ gravity, where G is Gauss-Bonnet invariant. Here, by employing the hyperbolic hybrid scale factor $a = e^{mt}\left(sinh\,(t)\right)^{n}$ in which m, n are positive constants,withthe power-law$f(G) = \beta G^{m+1}$ model where β, m are arbitrary constants, we computed the physical and geometrical properties of the cosmological parameters of the model. The results of the model parameters are well satisfied with recent cosmological observational data. To get the exact solution of the field equation of the Bianchi type III model in the presence of an anisotropic dark fluid, we consider the relation in which the shear scalar$(\sigma)$is proportional to the expansion scalar(θ), resulting in $C = A^{n}$. Furthermore, the energy conditions for the power-law $f(G)$ model are graphically examined, and it turns out that the null energy condition (NEC), weak energy condition (WEC), and dominant energy condition (DEC) are well satisfied except for the strong energy condition (SEC). The violation of the strong energy conditions (SEC) indicates the $f(G)$ model supports the universe's current expansion with negative pressure, having a quintessence model in the present and Λ cold dark matter (CDM) model in the future.

Keywords: Anisotropic universe, Bianchi type III universe, Dark-energy, Energy conditions, Quintessence model, $f(G)$gravity.

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1. Introduction

A general theory of relativity is a straightforward explanation of the cosmos that is also known as Einstein's theory of relativity. Although this theory has achieved remarkable success in modern physics, it eventually falls short of explaining the nature of the universe in terms of the current cosmic acceleration. As specific observational sources have already noted, the cosmos expands faster than expected due to the presence of "dark energy," an odd element with negative pressure. The

Equation of State parameter (EoS) can be used to distinguish the properties of dark energy. It is stated as $\omega = \frac{p}{\rho}$ where p and ρ represent the pressure and energy density of dark energy, respectively.

As an alternative theory of dark energy, Nojiri and Odintsov [1] introduced the modified Gauss-Bonnet gravity, or $f(G)$ gravity, where $f(G)$ is a generic function of the Gauss-Bonnet (GB) invariant G. The invariant G takes the form $G \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ and it is a 4-dimensional topological invariant. Under modified Gauss-Bonnet or $f(G)$ gravity, different classes of dark energy cosmologies are investigated [2], which can be seen as motivated by string considerations. The gravitation technique, which can predict the acceleration of the late-time universe with a natural transition from deceleration to acceleration [3], takes the evolution of the universe into account. Sometimes, the cosmologically viable $f(G)$ models can cross the phantom divide line before reaching the current universe [4], and the model transits from the decelerating to accelerated phase and passes through the point $G = 0$ in which the differentiability $f(G)$ cannot admit the decelerating power-law and accelerating solution [5]. The $f(G)$ gravity theory can explain the current Λ cold dark matter (CDM) model of the universe without any other component; it can describe dark energy contribution and inflationary epoch [6].Suppose we look into the contribution of the Gauss-Bonnet (GB) term in $f(G)$ gravity. In that case, the term Gauss-Bonnet acts as a cosmological constant in an isotropic and homogeneous model filled with a holographic dark energy fluid in the linear and quadratic form of $f(G)$ model[7].Also, another possible explanation for the universe's current accelerated expansion is the correspondence phenomena between various dark energy models within modified gravity. In consideration of this cosmic phenomenon, Jawad, et al. [8] created a power-law solution that established a relationship between $f(G)$ gravity and holographic dark energy (HDE), and they also used a new-age graphic dark energy (NADE) model to study $f(G)$ gravity in this perspective [9]. Results from the new-age graphic dark energy $f(G)$ model and the holographic dark energy $f(G)$ model indicate that the current universe is expanding at an accelerated rate while maintaining instability, indicating the model will remain unstable as the universe evolves. Nevertheless, as compared to the regular HDE model, the reconstruction of $f(G)$ gravity using the ordinary and entropy-corrected (m, n) type holographic dark energy model appears more stable and realistic [10]. Yet again, Jawad, et al. [11] have expanded on their research by considering a range of scale factor choices to reconstruct the $f(G)$ model with HDE. As a result, the cosmic implications of the holographically reconstructed $f(G)$ gravity have been examined. By using a correspondence scenario for interacting and non-interacting schemes, Sharif and Saba [12] created a ghost dark energy $f(G)$ model in a homogeneous, isotropic universe with pressureless matter and a power-law scale factor.

This paper focuses on the study of $f(G)$ gravity with an anisotropic backdrop. It is well recognized that using isotropic models is one of the best ways to examine the large-scale framework of the universe. In addition, cosmological evidence like the cosmic microwave background (CMB) radiation suggests that the universe as it exists now is isotropic. However, according to specific ideas, the early universe might not have been entirely uniform, and the Bianchi model is the most basic model with an anisotropic background. The flat Friedmann-Lemaitre-Robertson-Walker (FLRW) model is thought to be a generalization of the Bianchi type I model, considered to be the most basic of the Bianchi models. Previous work has taken into consideration the Bianchi type I model in the context of $f(G)$ gravity theory. While Fayaz, et al. [13] demonstrated that the Bianchi type I power-law solution is limited to a particular class of $f(G)$ models, Sharif and Fatima [14] examined the energy conditions for two distinct model choices. In their discussion of the Bianchi type I model in $f(G)$ gravity, Hatwar, et al. [15] noted that around zero pressure and temperature, quark matter is converted into strange quark matter under the power-law and exponential-law models. According to Shamir [16] investigation, the anisotropic cosmological model, when combined with the Bianchi type I model, exhibits a singularity-free solution. The solution derived by exponential-law asymptotically represents the universe as the de-sitter space corresponding to an accelerated expansion of the universe. Additionally, using the same line element, researchers in $f(G)$ gravity study the dark energy cosmological model[17], the holographic dark energy cosmological model [18], and massless and massive scalar fields [19].

In this study, inspired by the work of the Bianchi type I model, we will investigate the Bianchi type III model in $f(G)$ gravity with a unique form of hyperbolic hybrid scale factor. The significance of the Bianchi type III model in cosmology lies in its ability to offer a structured approach to exploring the consequences of anisotropy within the universe. The Bianchi type III model of the universe has already been studied in the context of Einstein's and modified theories of gravity. In the case of power-law and exponential-law in Einstein's theory of gravity, the Bianchi type III model of the universe can eventually approach isotropy even when dark fluid is present [20]. However, the Bianchi type III universe model remains anisotropic in the presence of dark fluid throughout the universe's evolution in the modified theory of gravity known as $f(R)$ gravity, as studied by Sharif and Kausar [21]. The same outcome is observed in Lyra's geometry in the presence of massive scalar fields with variable deceleration parameters [22]. Moreover, Tiwari, et al. [23] discovered that in the Bianchi type III model, the equation of state parameter (ω) filled with barotropic and dark energy ends up with quintessence dark energy. Korunur [24] showed that in the Bianchi type III model, the tsallis holographic dark energy's deceleration parameter changes from early deceleration to late-time cosmic acceleration with scalar fields.

The discussion that occurred above has motivated us to study the dynamics of an anisotropic, spatially homogeneous Bianchi type III model in $f(G)$ gravity. The objective of this work is to determine the exact solution of the field equation of the Bianchi type III model with the power-law $f(G)$ model, considering the hyperbolic hybrid scale factor in the presence of dark energy fluid. To achieve this, we adhere to the following paper structure: Sect. 2 provides a brief introduction to field equations in $f(G)$ gravity; Sect. 3 discusses solutions to the field equations for a particular $f(G)$ model choice. Section 4 discusses the results and the discussion. Section 5 presents the papers' concluding remarks.

2. Field Equations in $f(G)$Gravity

The action of modified Gauss-Bonnet gravity [25] is:

$$S = \frac{1}{2k}\int d^4x\sqrt{-g}[R + f(G)] + S_M(g^{\mu\nu},\psi) \qquad (1)$$

Where ψ indicates the matter fields, k is the coupling constant, g is the determinant of the metric tensor $g_{\mu\nu}$, and $S_M(g^{\mu\nu},\psi)$ is the matter action, in which matter is minimally linked to the metric tensor. It appears that $f(G)$ gravity is a purely metric theory of gravity because of this coupling of matter to the metric tensor.

The function $f(G)$ may represent any function of the GB invariant G:

$$G \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} \qquad (2)$$

Where R is the Ricci scalar and $R_{\mu\nu}$ and $R_{\mu\nu\sigma\rho}$ denote the Ricci and Riemann tensor. Gravitational field equations are obtained by varying the action in Equation 1 concerning the metric tensor.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 8\left[R_{\mu\rho\nu\sigma} + R_{\rho\nu}g_{\sigma\mu} - R_{\rho\sigma}g_{\nu\mu} - R_{\mu\nu}g_{\sigma\rho} + R_{\mu\sigma}g_{\nu\rho} + \frac{1}{2}R\left(g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho}\right)\right] \times \nabla^\rho\nabla^\sigma F + (Gf_G - f)g_{\mu\nu} = kT_{\mu\nu} \qquad (3)$$

Where the operator ∇_μ denotes the covariant derivative and f_G represents the derivative of f concerning G.

The line element for a spatially homogeneous, anisotropic, and Bianchi type III spacetime is given by

$$ds^2 = dt^2 - A^2dx^2 - B^2e^{-2\alpha x}dy^2 - C^2dz^2 \qquad (4)$$

Where A, B, and C are cosmic scale factors, and α is constant. Here, it should be noted that Equation 4 recovers the Bianchi type I model by setting the value $\alpha = 0$ from the Bianchi type III model.

The Ricci scalar and Gauss-Bonnet invariant are:

$$R = -2\left[\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} - \frac{\alpha^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC}\right] \qquad (5)$$

$$G = 8\left[\frac{\ddot{A}\dot{B}\dot{C}}{ABC} + \frac{\dot{A}\ddot{B}\dot{C}}{ABC} + \frac{\dot{A}\dot{B}\ddot{C}}{ABC} - \frac{\alpha^2}{A^2}\cdot\frac{\ddot{C}}{C}\right] \qquad (6)$$

Where the dot represents the derivative with respect to ‘t’.

Here, assuming the universe is filled with a dark energy fluid, the energy-momentum tensor become:

$$T_\nu^\mu = diag[\rho, -p_x, -p_y, -p_z] \qquad (7)$$

where ρ is the energy density of the fluid and p_x, p_y and p_z are the pressures along the x, y, and z axes. The fluid is characterized by the EoS $p = \omega\rho$, where ω is not necessarily constant.

From Equation 7, it follows that

$$T_\nu^\mu = diag[1, -\omega_x, -\omega_y, -\omega_z]\rho \qquad (8)$$

Where ω_x, ω_y and ω_z are the directional EoS parameter along the x, y, and z axes.

The deviation from isotropy can be obtained as

$$\omega_x = \omega,\ \omega_y = \omega + \delta \text{ and } \omega_z = \omega + \gamma$$

Where ω is the deviation-free EoS parameter of the fluid and the skewness parameters δ and γ are the deviation from ω along y and z axes.

In this case, the energy-momentum tensor becomes:

$$T_\nu^\mu = diag[1, -\omega, -(\omega + \delta), -(\omega + \gamma)]\rho \qquad (9)$$

The average scale factor $a(t)$ and the volume scale factor V is defined by:

$$V = a^3 = ABC \qquad (10)$$

The average Hubble parameter H, expansion scalar θ, and deceleration parameter q are defined by:

$$H = \frac{\dot{a}}{a} = \frac{1}{3}(H_x + H_y + H_z) \qquad (11)$$

Where, $H_x = \frac{\dot{A}}{A}, H_y = \frac{\dot{B}}{B}, H_z = \frac{\dot{C}}{C}$ are directional Hubble parameters along x, y, and z axes, respectively.

$$\theta = 3H = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \qquad (12)$$

$$q = -1 + \frac{d}{dt}\left(\frac{1}{H}\right) \qquad (13)$$

The deceleration parameter q is the most important quantity that can measure the universe's expansion rate depending on the sign of q, and SNe Ia observation data also claimed that if the value of q is in between 0 to -1, i.e., $-1 < q < 0$ that shows the present expansion of the universe is accelerating. Here, if $q > 0$ indicates expansion is deflation, while $q < 0$ indicates expansion is inflation. However, $q = 0$, the universe's expansion rate is constant. Therefore, with the help of the deceleration parameter, we can easily determine the states of the universe, whether it's inflation or deflation.

The mean anisotropy parameter A_m and shear scalar σ^2 are defined by

$$A_m = \frac{1}{3}\sum\left(\frac{\Delta H_i}{H}\right)^2;\ i = x, y, z \qquad (14)$$

Where $\Delta H_i = H_i - H;\ i = x, y, z$

$$\sigma^2 = \frac{3}{2}A_mH^2 \qquad (15)$$

Now, using Equations 4 and 9, the field Equation 3 takes the form:

$$-\frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC} + 8\left(\frac{\ddot{B}\dot{C}}{BC} + \frac{\dot{B}\ddot{C}}{BC}\right)\dot{f}_G + 8\frac{\dot{B}\dot{C}}{BC}\ddot{f}_G - Gf_G + f = \kappa\omega\rho \qquad (16)$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} - \frac{\dot{A}\dot{C}}{AC} + 8\left(\frac{\ddot{A}\dot{C}}{AC} + \frac{\dot{A}\ddot{C}}{AC}\right)\dot{f}_G + 8\frac{\dot{A}\dot{C}}{AC}\ddot{f}_G - Gf_G + f = \kappa(\omega + \delta)\rho \qquad (17)$$

$$-\frac{\ddot{A}}{A}-\frac{\ddot{B}}{B}-\frac{\dot{A}\dot{B}}{AB}+\frac{\alpha^2}{A^2}+8\left(\frac{\dot{A}\ddot{B}}{AB}+\frac{\ddot{A}\dot{B}}{AB}\right)\dot{f}_G+8\left(\frac{\dot{A}\dot{B}}{AB}-\frac{\alpha^2}{A^2}\right)\ddot{f}_G-Gf_G+f=\kappa(\omega+\gamma)\rho \quad (18)$$

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}-\frac{\alpha^2}{A^2}-24\frac{\dot{A}\dot{B}\dot{C}}{ABC}\dot{f}_G+8\frac{\alpha^2}{A^2}\cdot\frac{\dot{C}}{C}\dot{f}_G+Gf_G-f=\kappa\rho \quad (19)$$

$$\alpha\left(\frac{\dot{A}}{A}-\frac{\dot{B}}{B}\right)=0 \quad (20)$$

These are complicated and highly non-linear differential equations.

From Equation 20, we get:

$$B=c_1A \quad (21)$$

Here c_1 is the integration constant. Without loss of generality, we can consider $c_1=1$.

Now, Equation 21 can be written as

$$B=A \quad (22)$$

Consequently, the directional EoS parameters ω_x and ω_y along xy axes become identical. So, in this case, the energy-momentum tensor becomes

$$T_\nu^\mu=diag[1,-\omega,-\omega,-(\omega+\gamma)]\rho \quad (23)$$

With the relation (22), the field Equations 16, 17, 18, and 19 now take the form

$$-\frac{\ddot{A}}{A}-\frac{\ddot{C}}{C}-\frac{\dot{A}\dot{C}}{AC}+8\left(\frac{\dot{A}\ddot{C}}{AC}+\frac{\ddot{A}\dot{C}}{AC}\right)\dot{f}_G+8\frac{\dot{A}\dot{C}}{AC}\ddot{f}_G-Gf_G+f=\kappa\omega\rho \quad (24)$$

$$-2\frac{\ddot{A}}{A}-\frac{\dot{A}^2}{A^2}+\frac{\alpha^2}{A^2}+16\frac{\dot{A}\ddot{A}}{A^2}\dot{f}_G+8\left(\frac{\dot{A}^2}{A^2}-\frac{\alpha^2}{A^2}\right)\ddot{f}_G-Gf_G+f=\kappa(\omega+\gamma)\rho \quad (25)$$

$$\frac{\dot{A}^2}{A^2}+2\frac{\dot{A}\dot{C}}{AC}-\frac{\alpha^2}{A^2}+8\left(\frac{\alpha^2}{A^2}\cdot\frac{\dot{C}}{C}-3\frac{\dot{A}^2\dot{C}}{A^2C}\right)\dot{f}_G+Gf_G-f=\kappa\rho \quad (26)$$

Here, we use a physical condition that the shear scalar σ and the expansion scalar θ are proportional to each other $(\sigma\propto\theta)$, which leads to

$$C=A^n \quad (27)$$

The above condition (27) has already been used in the literature [26-28] to obtain the precise solution of the field equation.

Where n is an arbitrary real number and $n\neq 0,1$ for non-trivial solutions.

Thus, with Equation 27, the field Equations 24, 25, and 26 reduces to:

$$-(n+1)\frac{\ddot{A}}{A}-n^2\frac{\dot{A}^2}{A^2}+8\left(2n\frac{\dot{A}\ddot{A}}{A^2}+n(n-1)\frac{\dot{A}^3}{A^3}\right)\dot{f}_G+8n\frac{\dot{A}^2}{A^2}\ddot{f}_G-Gf_G+f=\kappa\omega\rho \quad (28)$$

$$-2\frac{\ddot{A}}{A}-\frac{\dot{A}^2}{A^2}+\frac{\alpha^2}{A^2}+16\frac{\dot{A}\ddot{A}}{A^2}\dot{f}_G+8\left(\frac{\dot{A}^2}{A^2}-\frac{\alpha^2}{A^2}\right)\ddot{f}_G-Gf_G+f=\kappa(\omega+\gamma)\rho \quad (29)$$

$$(1+2n)\frac{\dot{A}^2}{A^2}-\frac{\alpha^2}{A^2}+8n\left(\frac{\alpha^2}{A^2}\cdot\frac{\dot{A}}{A}-3\frac{\dot{A}^3}{A^3}\right)\dot{f}_G+Gf_G-f=\kappa\rho \quad (30)$$

This is a system of three differential equations in four unknowns A,ρ,ω,γ. Hence, to find a determinate solution to Equations 28-30, we adopt the following scale factor, which is a combination of exponential and hyperbolic functions known as the hyperbolic hybrid scale factor [29]:

$$a=e^{mt}\left(sinh(t)\right)^n \quad (31)$$

Where m,n are positive constants.

Here, both the exponential and hyperbolic functions are dependent on time, admitting a point-type singularity in which the model begins to expand with the BigBang for the time $t=0$. Esmaeili [30] mentioned the role of scaling constants, which is vital if the parametrization scale factor is in the form of hyperbolic. In the Bianchi type V model, Mishra, et al. [31] found the universe with hyperbolic scale factor transits from early cosmic deceleration to late cosmic acceleration. With the exponential scale factor, the universe shows an accelerated expansion of the universe at a late time.

After using Equation 31 in Equation 10, metric potential becomes

$$A=B=\left[e^{mt}\left(sinh(t)\right)^n\right]^{3/(n+2)} \quad (32)$$

$$C=\left[e^{mt}\left(sinh(t)\right)^n\right]^{3n/(n+2)} \quad (33)$$

By the result of Equations 32 and 33, the directional Hubble parameter H_x,H_yH_z can be obtained as

$$H_x=H_y=\frac{3(msinh(t)+ncosh(t))}{(n+2)sinh(t)} \text{ and } H_z=\frac{3n(msinh(t)+ncosh(t))}{(n+2)sinh(t)} \quad (34)$$

So, model (4) with Equations 32 and 33 takes the form

$$ds^2=dt^2-\left[e^{mt}\left(sinh(t)\right)^n\right]^{6/(n+2)}dx^2-\left[e^{mt}\left(sinh(t)\right)^n\right]^{6/(n+2)}e^{-2\alpha x}dy^2-\left[e^{mt}\left(sinh(t)\right)^n\right]^{6n/(n+2)}dz^2 \quad (35)$$

The Ricci scalar and the Gauss-Bonnet invariant are

$$R=\frac{6n}{(sinh(t))^2}-\frac{18(n^2+2n+3)(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2}+\frac{2\alpha^2}{\left(e^{mt}(sinh(t))^n\right)^{6/(n+2)}} \quad (36)$$

$$G=\frac{24A^2n^2\left(e^{mt}(sinh(t))^n\right)^{-6/(n+2)}}{(n+2)(sinh(t))^2}-\frac{144A^2n\left(e^{mt}(sinh(t))^n\right)^{-6/(n+2)}(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2}-\frac{648n\left(n-(msinh(t)+ncosh(t))^2\right)(msinh(t)+ncosh(t))^2}{(n+2)^3(sinh(t))^4} \quad (37)$$

The volume, Hubble parameter, expansion scalar, deceleration parameter, mean anisotropy, and shear scalar are:

$$V = \left[e^{mt}\left(sinh(t)\right)^{n}\right]^{3} \quad (38)$$

$$H = \frac{msinh(t)+ncosh(t)}{sinh(t)} \quad (39)$$

$$\theta = 3\left(\frac{msinh(t)+ncosh(t)}{sinh(t)}\right) \quad (40)$$

$$q = -1 + \frac{n}{\left(msinh(t)+ncosh(t)\right)^{2}} \quad (41)$$

$$A_m = \frac{2(n-1)^2}{(n+2)^2} = constant \quad (42)$$

$$\sigma^2 = \frac{3(n-1)^2\left(msinh(t)+ncosh(t)\right)^2}{(n+2)^2\left(sinh(t)\right)^2} \quad (43)$$

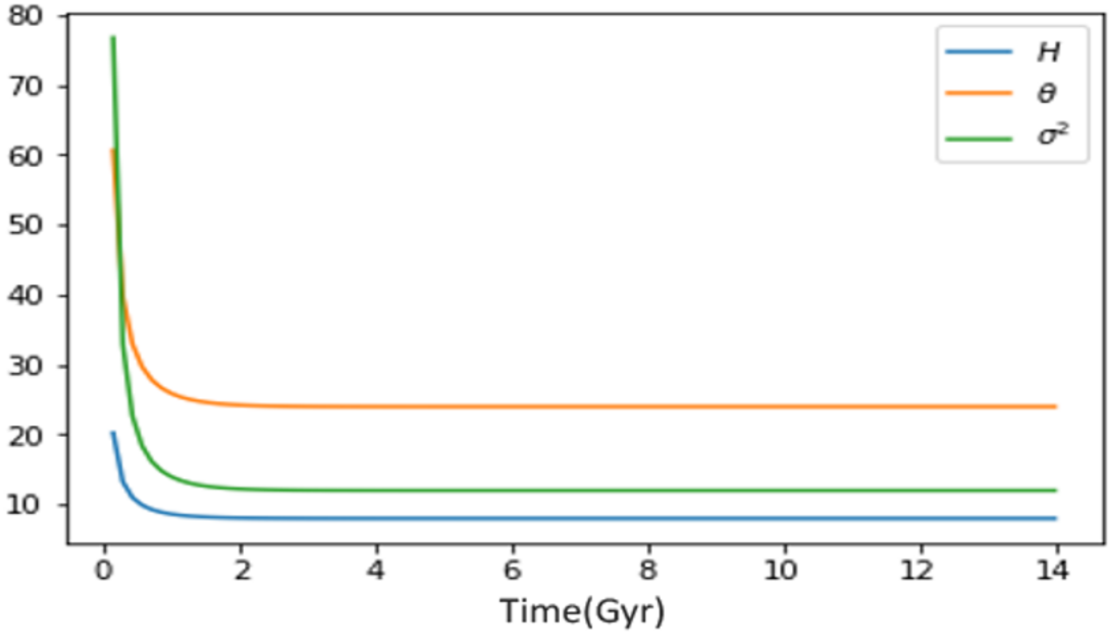

Figure 1. Hubble parameter(H), Expansion scalar(θ), and Shear scalar (σ^2) vs. time (Gyr).

Here, Figure 1 illustrates the Hubble parameter, Expansion scalar, and shear scalar as a function of time $'t'$ showcasing initially massive, which means $H, \theta, \sigma^2 \to \infty$ when $t \to 0$ and at $t \to \infty$, the model's parameters converge to a constant value. From Equation 38, it is observed that thevolume is zero with $t = 0$, and the model is expanding with $V \to \infty$ when $t \to \infty$. The positive value of the Hubble parameter $(H > 0)$ indicates the model is expanding, whereas a negative value $(H < 0)$ shows the model is contracting. At $t = 90$, the Hubble parameter yields results$H = m$ which match the result of Shaikh, et al. [18] in which the Hubble parameter is dependent on constant $'m'$ and consequently $\frac{dH}{dt} = 0$ implies the fastest rate of expansion of the universe and the most significant value of Hubble's parameter. Equation 42 shows that the mean anisotropy parameter A_m depends on constant $'n'$ and $A_m \neq 0$ for $n \neq 1$while $A_m = 0$ for$n = 1$. Therefore, the model of the universe reflects expanding anisotropically for $n \neq 1$ while the model is isotropic for$n = 1$.

3. Bianchi Type-III Universe with Power-Law $f(G)$ Model

Considering the power-law $f(G)$ model [2] with:

$$f(G) = \beta G^{m+1} \quad (44)$$

Where β and m are unknown constants.

According to Nojiri, et al. [32], these power-law $f(G)$ models, which predict early-time inflation and late-time acceleration, are consistent with the observational evidence. If the second derivative of $f(G)$ with respect to G is divergence at $G = 0$, then the power-law $f(G)$ is not cosmologically viable, according to De Felice and Tsujikawa [33]. Furthermore, taking into account the previously mentioned power-law $f(G)$ model, Shamir investigated about anisotropic [16] and dark energy [17] cosmological models for spatially homogenous, anisotropic, locally rotationally symmetric (LRS) Bianchi type-I line elements. Using the same line element, Shaikh, et al. [18]investigated three distinct forms of expansion laws (volumetric, power-law, and hybrid) for the holographic dark energy model.

From Equation 44, it follows that

$$f_G(G) = \beta(m+1)G^m \tag{45}$$

Here, we choose $\beta = {}^{1}/_{(m+1)}$ for further analysis. From Equation 45, we may obtain:

$$\dot{f_G} = mG^{m-1}\dot{G}, \qquad \ddot{f_G} = m\left(G^{m-1}\ddot{G} + (m-1)G^{m-2}\dot{G}^2\right) \tag{46}$$

Where $G = \frac{24A^2n^2\left(e^{mt}(sinh(t))^n\right)^{-6/(n+2)}}{(n+2)(sinh(t))^2} - \frac{144A^2n\left(e^{mt}(sinh(t))^n\right)^{-6/(n+2)}(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} - \frac{648n\left(n-(msinh(t)+ncosh(t))^2\right)(msinh(t)+ncosh(t))^2}{(n+2)^3(sinh(t))^4}$

Now, using Equations 32, 37, 44, 45, and 46, we get the expression of energy density and pressure along x, z-axes as follows.

$$\rho = \frac{1}{\kappa}\left[\frac{9(2n+1)(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} - \frac{\alpha^2}{\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}} + 8n\left(\frac{3\alpha^2(msinh(t)+ncosh(t))}{(n+2)sinh(t)\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}} - \frac{81(msinh(t)+ncosh(t))^3}{(n+2)^3(sinh(t))^3}\right)\dot{f_G} + Gf_G - f\right] \tag{47}$$

$$p_x = p_y = \frac{1}{\kappa}\left[\frac{3n(n+1)}{(n+2)(sinh(t))^2} - \frac{9(n^2+n+1)(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} + 8n\left(\frac{-18n(msinh(t)+ncosh(t))}{(n+2)^2(sinh(t))^3} + \frac{27(n+1)(msinh(t)+ncosh(t))^3}{(n+2)^3(sinh(t))^3}\right)\dot{f_G} + \frac{72n(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2}\ddot{f_G} - Gf_G + f\right] \tag{48}$$

$$p_z = \frac{1}{\kappa}\left[\frac{6n}{(n+2)(sinh(t))^2} - \frac{27(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} + \frac{\alpha^2}{\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}} + \left(\frac{432(msinh(t)+ncosh(t))^3}{(n+2)^3(sinh(t))^3} - \frac{144n(msinh(t)+ncosh(t))}{(n+2)^2(sinh(t))^3}\right)\dot{f_G} + \left(\frac{72(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} - \frac{8\alpha^2}{\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}}\right)\ddot{f_G} - Gf_G + f\right] \tag{49}$$

The expression of the Equation of state parameter (EoS) is defined as $\omega = \frac{p}{\rho}$, so from Equations 47 and 48, it becomes

$$\omega = \left[\frac{3n(n+1)}{(n+2)(sinh(t))^2} - \frac{9(n^2+n+1)(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} + 8n\left(\frac{-18n(msinh(t)+ncosh(t))}{(n+2)^2(sinh(t))^3} + \frac{27(n+1)(msinh(t)+ncosh(t))^3}{(n+2)^3(sinh(t))^3}\right)\dot{f_G} + \frac{72n(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2}\ddot{f_G} - Gf_G + f\right] \Big/ \left[\frac{9(2n+1)(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} - \frac{\alpha^2}{\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}} + 8n\left(\frac{3\alpha^2(msinh(t)+ncosh(t))}{(n+2)sinh(t)\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}} - \frac{81(msinh(t)+ncosh(t))^3}{(n+2)^3(sinh(t))^3}\right)\dot{f_G} + Gf_G - f\right] \tag{50}$$

The skewness parameter γ can describe the amount of anisotropy in the dark energy fluid, and for this model, we get

$$\gamma = \left[-\frac{3n(n-1)}{(n+2)(sinh(t))^2} + \frac{9(n^2+n-2)(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} + \frac{\alpha^2}{\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}} - 8(n-1)\left(-\frac{18n(msinh(t)+ncosh(t))}{(n+2)^2(sinh(t))^3} + \frac{27(msinh(t)+ncosh(t))^3}{(n+2)^2(sinh(t))^3}\right)\dot{f_G} - 8\left(\frac{9(n-1)(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} + \frac{\alpha^2}{\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}}\right]\ddot{f_G} \Big/ \left[\frac{9(2n+1)(msinh(t)+ncosh(t))^2}{(n+2)^2(sinh(t))^2} - \frac{\alpha^2}{\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}} + 8n\left(\frac{3\alpha^2(msinh(t)+ncosh(t))}{(n+2)sinh(t)\left(e^{mt}(\sinh(t))^n\right)^{6/(n+2)}} - \frac{81(msinh(t)+ncosh(t))^3}{(n+2)^3(sinh(t))^3}\right)\dot{f_G} + Gf_G - f\right] \tag{51}$$

3.1. Energy Conditions

Energy conditions generated from the Raychaudhuri equation play a significant role in the modified cosmological model. Sharif and Fatima [14]; Shamir [17]; Bamba, et al. [34], and García, et al. [35] have presented the energy condition in $f(G)$ gravity to assess the model's viability and provide some significant insight in their work. The energy conditions are a crucial tool to describe the singularity problems of spacetime and explain the nature of null, time-like, or light-like geodesics. Generally, the energy conditions are classified as (i) the null energy condition (NEC), (ii) the weak energy condition (WEC), (iii) the strong energy condition (SEC), and (iv) the dominant energy condition (DEC) and can be expressed as follows:

(1) NEC: $\rho + p_x \geq 0, \rho + p_z \geq 0$
(2) WEC: $\rho \geq 0, \rho + p_x \geq 0, \rho + p_z \geq 0$
(3) SEC: $\rho + 3p_x \geq 0, \rho + 3p_z \geq 0, \rho + p_x \geq 0, \rho + p_z \geq 0$
(4) DEC: $\rho \geq 0, \rho \pm p_x \geq 0, \rho \pm p_z \geq 0$

We also analyze these energy conditions for the given power-law $f(G)$ model.

3.2. Statefinder Parameter

The statefinder parameter $\{r, s\}$ proposed by Sahni, et al. [36] can distinguish features of the dark energy models and is defined as

$$r = \frac{\dddot{a}}{aH^3} = 1 + \frac{3\dot{H}}{H^2} + \frac{\ddot{H}}{H^3}, \quad (52)$$

$$s = \frac{r-1}{3\left(q - \frac{1}{2}\right)} = \frac{-2(3H\dot{H} + \ddot{H})}{3H(3H^2 + 2\dot{H})} \quad (53)$$

Here H is the Hubble parameter, and the dot represents the differentiation with respect to time 't'. With Equation 39, the statefinder pair $\{r, s\}$ can take the form

$$r = \frac{(m^3 - 3mn)\left(sinh(t)\right)^3 + n(n^2 - 3n + 2)\left(cosh(t)\right)^3}{\left(msinh(t) + ncosh(t)\right)^3} + \frac{n(3m^2 - 3n + 2)\left(sinh(t)\right)^2 cosh(t)}{\left(msinh(t) + ncosh(t)\right)^3} + \frac{3mn(n-1)sinh(t)\left(cosh(t)\right)^2}{\left(msinh(t) + ncosh(t)\right)^3}$$

$$s = \frac{4n(3n-2)(cosh(t))^3 + 12mnsinh(t)(cosh(t))^2 + 2n(2-3n)cosh(t) - 6mnsinh(t)}{3(msinh(t) + ncosh(t))(2n - 4n(cosh(t))^2 + 3m^2(sinh(t))^2 + 3n^2(cosh(t))^2 + 6mnsinh(t)cosh(t))}$$

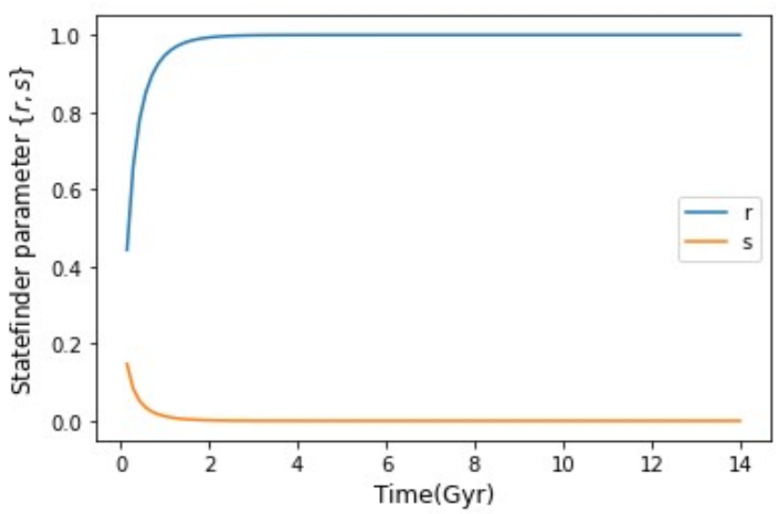

Figure 2. Statefinder parameter $\{r, s\}$ vs. time (Gyr).

From Figure 2, it is seen that the model initially $r < 1, s > 0$ indicates quintessence and phantom dark energy, and at $t \to \infty$, we get the pair $\{r, s\} \to \{1,0\}$ which shows that the model corresponds to Λ CDM model in future.

4. Results and Discussion

In this paper, we investigated cosmological models consistent with the $f(G)$ gravity theory. The models discussed in this study have the following prominent features:

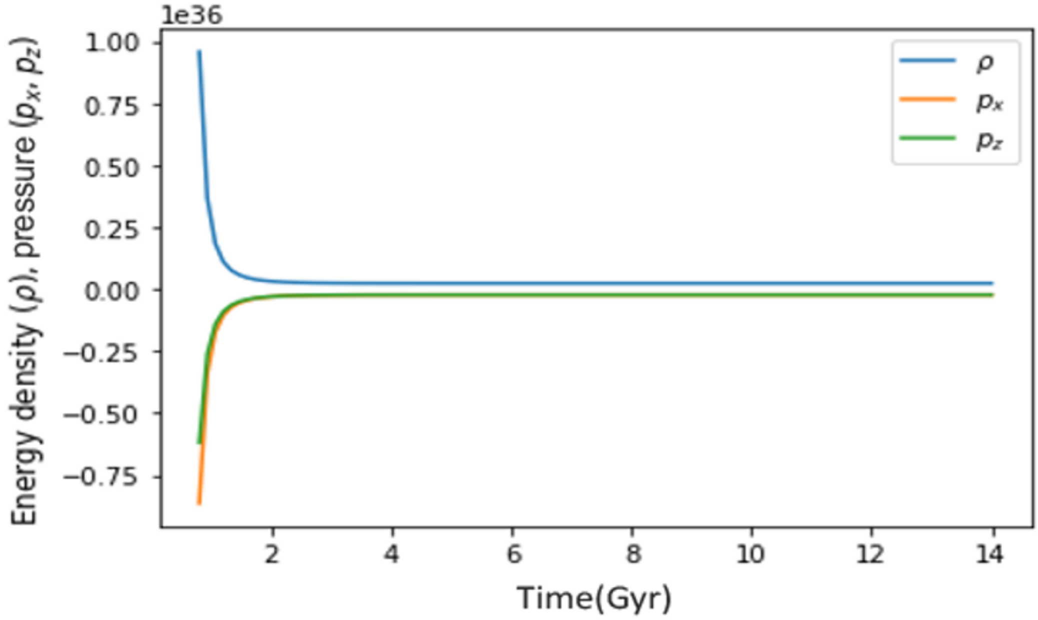

Figure 3. Energy density(ρ), Pressure along x,z- axes(p_x, p_z) vs. time (Gyr).

Equations 47, 48, and 49, which describe the behavior of energy density and pressure along the x and z axes, offer a consistent solution for the parametric values $m = 6, n = 2$, and $\alpha = 0.02$. As depicted in Figure 3, the energy density starts relatively high and gradually reduces as time passes; however, it remains in a favorable region throughout the evolution. It is also observed that the value $m > 1, n = 1,2,3$, and $\alpha = 0.02$ the energy density are always positive and converge to zero at a later time. On the other hand, the result obtained in $f(R, T)$ gravity for LRS Bianchi type I [29] is quite different compared to ours. The evolution of pressure along the x, z-axes is negative for $0 < t < 14$, and eventually, as time progresses, the evolution of pressure approaches constant value for both x, z axes.

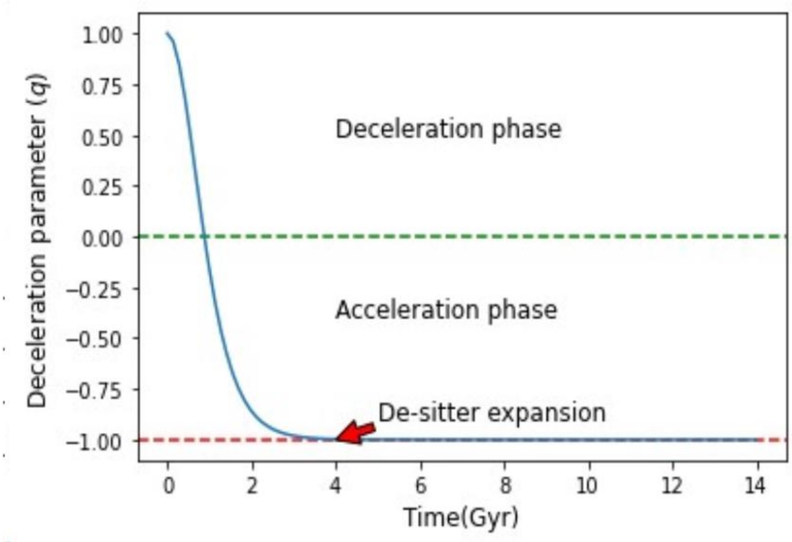

Figure 4. Deceleration parameter(q) vs. time(Gyr).

The deceleration parameter $q = -1 + \frac{n}{(m sinh(t) + n cosh(t))^2}$ exhibits sign-flipping behavior from early deceleration to present acceleration depending on the value of $'n'$ as depicted in Figure 4. It is seen that the deceleration parameter $q = -1 + \frac{1}{n} > 0$ for $0 < n < 1$ and when the value of $'n'$ rise up $(n > 1)$ falls with the exponential expansion of the universe $(-1 < q < 0)$[29] and setting up with de-sitter expansion $q = -1$ at $t \to \infty$. Tiwari, et al. [37] presented the decelerating and accelerating phase of the universe dependent on constant value $'n'$ with Bianchi type III string cosmological model. They mentioned the present value of the universe $q_0 = -0.92$. From Equation 41, we can also have $q_0 \sim 0.99$ for $t_0 = 13.8$ Gyr(Gigayear), and, here t_0 ,q_0 are the present time and present value of the deceleration parameter. Therefore, the value of the deceleration parameter is consistent with the current observational data.

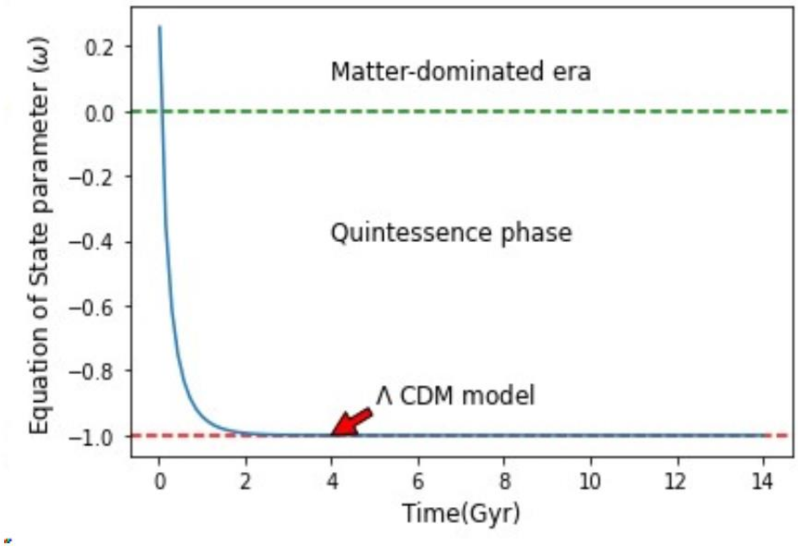

Figure 5. Equation of State parameter (ω) vs. time(Gyr).

The EoS parameter, ω shownin Figure 5, is one of the most important parameters that can be used to interpret the different regimes of the universe. Figure 5 shows that the universe transits from the early decelerated phase to the current accelerated phase depending on time 't'. Initially, the EoS parameter is dominated by the matter-dominated era $(0 < \omega < 1/3)$ dependingon the time $0 < t < 0.08$, and, later gradually stabilizes in the quintessence phase $(-1 < \omega < 0)$ for $0.08 < t < 14$, which corresponds to the ΛCDM model $(\omega = -1)$ at $t \to \infty$. Furthermore, the EoS parameter has been constrained by many observational data, which are WMAP+CMB, $\omega = -1.073^{+0.090}_{-0.089}$ [38] WMAP+Supernova, $\omega = -1.084 \pm 0.063$ [38] Supernova Cosmology Project, $\omega = -1.035^{+0.055}_{-0.059}$ [39] Planck 2018, $\omega = -1.03 \pm 0.03$ [40]. Here, for $t_0 = 13.8$ Gyr, we also have $\omega_0 \sim 0.99$ one compatible with the above observational data.

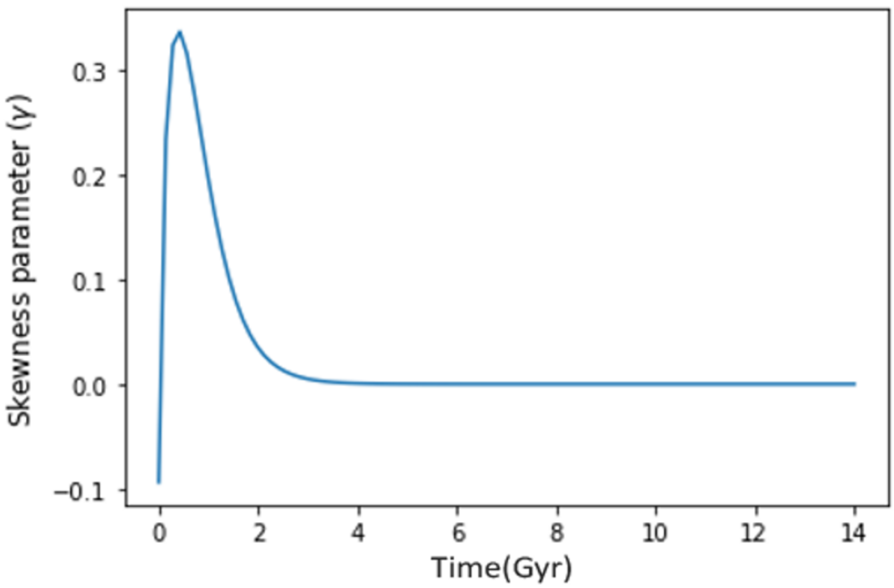

Figure 6. Skewness parameter (γ) vs. time(Gyr).

In the derived model, the evolutionary behavior of the skewness parameter γ is depicted in Figure 6, showing both positive and negative values for a significant time 't'. In literature Shamir [17]and Malik, et al. [41], similar behavior of the skewness parameter is noticeable based on the LRS Bianchi type I line element for a different value of ‘n’. In our present study, initially, it yielded a negative value for $0 < t < 0.02$, obtaining the parametric values $m = 6, n = 2$; later, it transformed into a positive value for $t \approx 0.03$. It is also pointed out that the skewness parameter γ reduces to an isotropic nature of the universe for $m = 0$ and $n = 1$ at the late time.

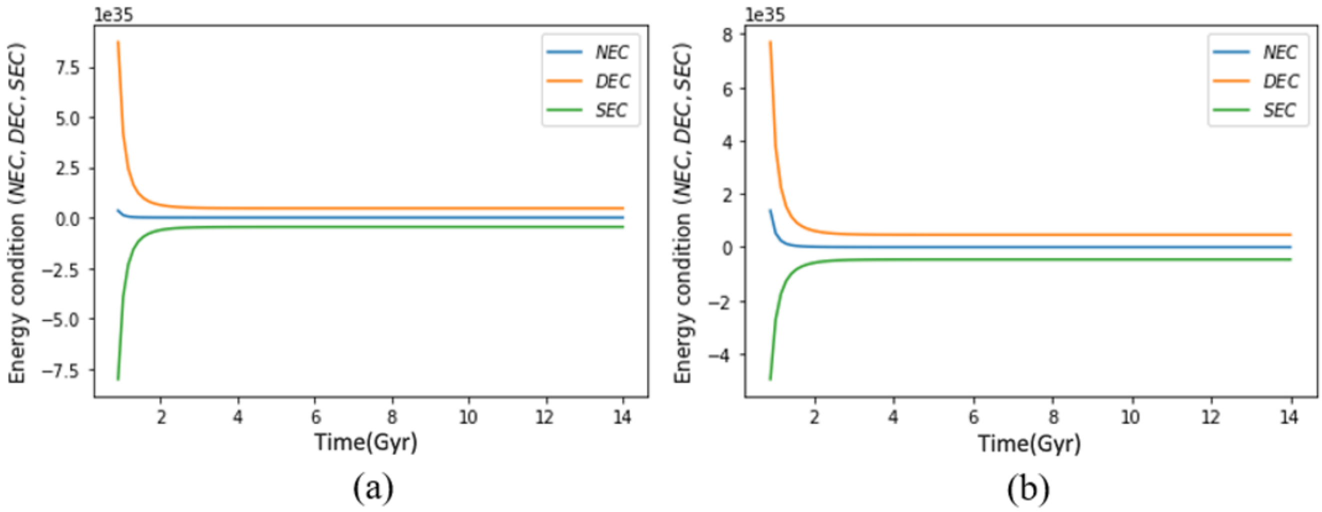

Figure 7. Energy condition (NEC, DEC, SEC) along x,z- axes vs. time(Gyr).

Figure 7 shows that the NEC, WEC, and DEC are well satisfied along the $x, z-$axes with violation of SEC against time't'. Here, to explain the present expansion of the universe, researchers focus more on the violation of SEC since the violation of SEC is the only energy condition responsible for the presence of dark energy with negative pressure. As a result, a violation of SEC can be seen in Figure 7; therefore, we can say that the model supports the universe's present expansion for $0 < t < 14$.

5. Conclusion

This work investigates an anisotropic and spatially homogenous Bianchi type III cosmological model with $f(G)$ gravity in the presence of an anisotropic dark energy fluid. The hyperbolic hybrid scale factor [29] is used to analyze the exact solution of the Bianchi type III model, and the power-law $f(G) = \beta G^{m+1}$model that was chosen for this purpose is compatible with observational data. The hyperbolic hybrid scale factor yields a sign-flipping deceleration parameter from early deceleration to present acceleration, where the parametric value 'n' is important. The model describes the expansion, shearing, and anisotropic nature of the universe throughout time, and the behavior of the model parameters is in good accord with the observational data. Throughout the entire cosmic development, the energy density is positive, which is consistent with the parametric values $m = 6, n = 2, \alpha = 0.02$. Additionally, the presence of dark energy is observed in the result of negative pressure$(x, z - axes)$, and at $t \to \infty$, the quintessence form of dark energy reaches to ΛCDM model [42].

In contrast, the statefinder parameter $\{r, s\}$ leads to the same result. The skewness parameter γ demonstrates the amount of anisotropy in the dark fluid, starting with a negative value, but for$t \approx 0.03$, the value γ tends to be positive. At $t \to \infty$ the value of $\gamma \neq 0$, therefore, the model of the universe always stays in the anisotropic phase and does not transform into an isotropy phase in the future.

Furthermore, in the case of $\gamma = 0$ the fluid's anisotropy is eliminated in the universe's future evolution [21, 43]. The energy conditions such as NEC, WEC, and DEC are satisfied with the entire evolution except the SEC, and the violation of the SEC represents the anisotropic universe in$f(G)$gravity that dominates the universe's current expansion. Therefore, we can conclude that the Bianchi type III cosmological model in $f(G)$ gravity substantially supports the universe's current expansion in the presence of anisotropic dark fluid, in which the model evolves with the quintessence model at the present and ΛCDM model in the future. Furthermore, more research can be carried out in $f(G)$ gravity to investigate the universe's current scenario and its evolution within the Bianchi type III model.

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