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Development of the schema for the differential of a function of several variables

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Abstract

The Purpose of this study aimed to analyze the understanding of the concept of the differential of a function of several variables among university students, adopting a constructivist and cognitive approach focused on the development of mental schemas. The research was conducted sequentially: 1) A theoretical analysis of the concept; 2) The design and implementation of specific instruction; 3) The collection and analysis of empirical data from questionnaires and interviews. The development of the differential scheme was characterized by the mental structures students used to relate its constituent elements. The schema's evolution through intra-, inter-, and trans-levels was evidenced by a process of synthesis and increased cognitive complexity. It is concluded that understanding the concept advances through the progressive evolution of the mental schema. The articulation of the intra, inter, and trans levels is fundamental for a complete grasp of the differential. The study supports designing didactic sequences based on the concept's genetic decomposition. This approach, aided by materials that facilitate transitions between representations, enhances the teaching of multivariable calculus.

Keywords: Understanding, differential, multivariate function, advanced mathematical thinking, genetic decomposition.

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1. Introduction

This research describes and explains how university-level students understand the concept of the differential of a function of several variables (DFSV). This is a property possessed by certain mathematical mappings with a domain in an open subset of the n-dimensional space $\mathbb{R}^n$ and a range in the m-dimensional space $\mathbb{R}^m$, which allows them to be approximated by a linear transformation at a point in their domain [1].

Regarding this, the phenomenon of understanding, based on the notion of schema development, is important in mathematics education within the field of teacher training. It helps identify aspects that require greater emphasis in teaching and provides guidelines on how to achieve this through the design of instructional activities and the development of educational materials validated through experimental testing [2].

2. Literature Review

2.1. Theorical Foundations

This research is framed within the context of Advanced Mathematical Thinking (AMT), which is defined as the cognitive phenomenon involving the interaction of mental processes used to represent, visualize, generalize, classify, conjecture, induce, analyze, synthesize, abstract, define, formalize, and prove mathematical ideas [3].

The study of understanding was based on the APOS Theory, an acronym denoting the mental structures of Action, Process, Object, and Schema. It is an epistemological framework rooted in Piaget's theory, which analyzes existing knowledge of a concept and its progression to a higher plane of thinking. At this level, knowledge is reorganized and reconstructed to form new structures through reflective abstraction, disequilibration, accommodation, assimilation, reflection, and the development of schemas [4].

Understanding is conceived as the execution of actions and processes on mental objects, and the subsequent construction and organization of these objects into schemas in order to use them for problem-solving. This process is made possible by the mental mechanism of reflective abstraction [5].

Regarding mental structures and their relationships, they are illustrated in Figure 1 and defined as follows:

Action: A transformation or manipulation of objects that the individual perceives as external. It is evidenced when the subject reacts to external stimuli with precise data on how to proceed at each step.

Process: a reflection and internalization of a sequence or repetition of actions. The subject can describe these actions and may even reverse the steps. A process is perceived as something internal and under the individual's control.

Object: Physical or mental entities resulting from the encapsulation of a process. This is evidenced when the subject reflects on the operations applied to the process, becomes aware of it as a whole, and performs transformations (actions or processes) on the process itself.

Schema: A collection of actions, processes, objects, and other previously developed schemas that constitute an individual's knowledge of a mathematical concept. Schema formation is a dynamic activity and a feedback-driven system [6].

The construction of these mental structures is made possible through the activation of the following mechanisms, which serve as examples of reflective abstraction:

Interiorization: The ability to use symbols, languages, images, and mental imagery to construct internal processes as ways of finding meaning in perceived phenomena [4].

Coordinating: The composition of two or more processes to construct a new action, process, or object [3].

Encapsulating: The conversion of a (dynamic) process into a (static) object [7].

Generalizing: The ability to apply a particular existing schema to a broader set of phenomena [4].

Reversing: Once a process exists internally, the subject can reverse it. This is not meant as simply undoing, but rather as a means of constructing a new process.

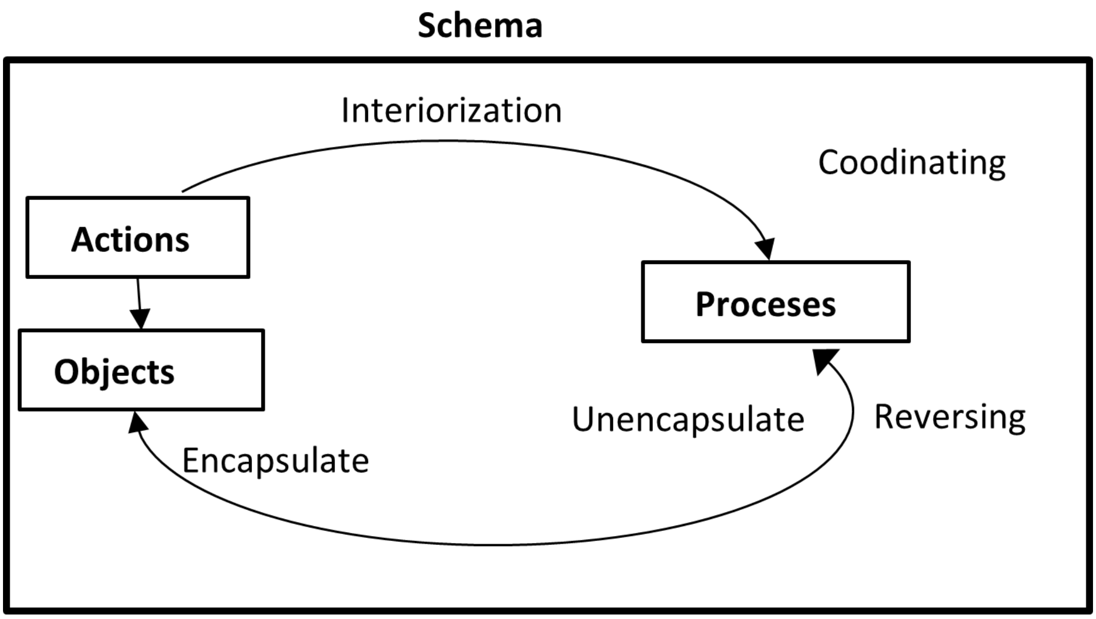

Figure 1.
Mental structures and mechanisms for constructing a Concept.
Source: Arnon, et al. [6].

Regarding the development of a schema, Piaget and García [4] establish the following levels:

Intra: "Is characterized by the discovery of any operative action and by the search for and analysis of its different internal properties or its immediate consequences. It is limited by the absence of coordination or relation of this pre-operation with others in an organized structure [6].

Inter. At this level, information of a similar nature begins to be grouped, and relationships between actions, processes, and objects start to be constructed [6].

Trans: Is acquired when a complete underlying structure is built, in which the relationships discovered at the inter level are understood, giving coherence to the schema [6].

The description of mathematics involved in the structure of a concept and how a subject might make the mental constructions that would lead to its understanding is called a genetic decomposition (GD) [6, 8]. Furthermore, it is not unique for a given concept [9].

2.2. Review of Relevant Studies

The review of research on the understanding of the concept of DFSV is structured around the following:

2.2.1. Relationship between Derivation and Differentiation

Research documents that activities focused on local linearity and the connection between the tangent plane and partial derivatives foster the understanding of partial and second derivatives [10]. These approaches allow for the analysis of personal meanings and mental constructions associated with these concepts [11].

Studies in the same vein explore how to establish relationships between the partial derivative, the directional derivative, and the differential. Results indicate that students can build these connections through explicit guidance that emphasizes the interpretation of the differential as a local approximation of the function, using genetic decompositions based on the APOS theoretical framework [11-14].

2.2.2. Teaching Difficulties and Intervention Strategies

It has been identified that some students tend to confuse the derivative of single-variable functions with partial derivatives in multivariable functions. To address this difficulty, didactic strategies are proposed that incorporate multiple representations (geometric, algebraic, and contextualized), mediated by software and visual tools. These interventions reduce the conceptual leap and allow for evaluating the impact of technological tools on learning outcomes [14, 15].

Additionally, to improve the interpretation of vector fields, instruction activities based on guided drawing have been designed, comparing their effectiveness with other representations [16, 17].

2.2.3. Contributions of Technological Tools

The use of dynamic tools and Computer Algebra Systems (CAS) in controlled environments facilitates the simultaneous integration of geometric, algebraic, and contextualized representations. When these tools are applied in didactic interventions focused on the local linear approximation of two-variable functions and their relationship with the tangent plane, partial derivatives, directional derivatives, gradient, and differential, they significantly enhance the geometric understanding of the differential concept and its interrelationships [10, 14, 15, 18].

2.2.4. Specific Studies on Cognitive Processes

Case study-based research has analyzed personal meanings and mental constructions for understanding partial and directional derivatives [12]. Concurrently, drawing analyses have been conducted to examine the relationship between visual attention, cognitive load, and performance on tasks related to vector fields [16].

The results obtained regarding the development of the schema are linked to the students' ability to relate the elements that constitute during the resolution of the questionnaire tasks and their reasoning in the interviews. These were organized into the Intra, Inter, and Trans triad [19].

3. Method

The research is qualitative in nature, focused on understanding, with the objective of describing and interpreting the educational reality from within Dorio, et al. [20]. A three-component cycle was implemented Figure 2, described as follows:

Theoretical Analysis: This involves the historical and epistemological study of the concept, as well as the examination of its presentation in textbooks. The aim is to establish and characterize the mathematical elements, the logical relationships that constitute the concept, and its forms of representation. The outcome of this component is the design of the GD.

Design and Implementation of Instruction: This is carried out through a cycle of Activities, Classes, and Exercises (ACE) designed to promote the mental constructions described in the GD.

Data Collection and Analysis: Information is gathered through questionnaires and semi-structured interviews. Its analysis is performed with reference to the GD, assessing the mental constructions observed.

The cycle is repeated until students demonstrate that they have constructed the mental processes and objects described in the GD and have achieved a understanding of the mathematical concept, allowing them to be placed at the different developmental levels based on the evidence [6].

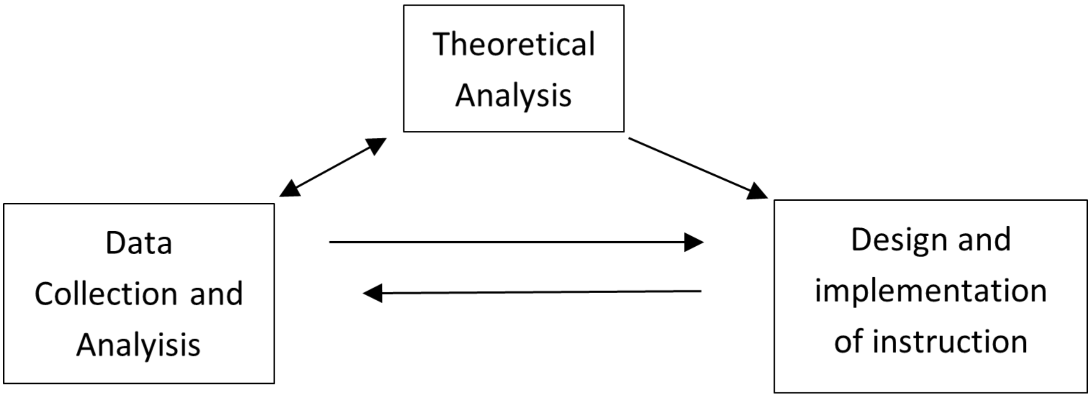

Figure 2.
The research cycle in APOS theory.

This research was conducted at the Universidad Pedagógica y Tecnológica de Colombia (UPTC) with groups of students from the undergraduate program in Mathematics. These students participated in the ACE cycle and took the course Multivariable Calculus and Real Analysis II. The participants' age range was 16 to 25 years old.

The iterative ACE cycle was carried out over a period of two and a half years, covering five academic semesters, during which the data collection instruments were designed, implemented, and refined. With the final group, consisting of nine students (pseudonyms E1 to E9). The analysis of this data yielded the results reported below.

4. Results

As part of the research cycle in the theoretical analysis phase, the mathematical elements that constitute the concept of the differential were identified. These are: the directional derivative (DD), the partial derivative (DP), the concept of a function differentiable at a point (FDE), and the differentials of: a real variable function (DIFR), a vector function of a real variable (DIFVR), a scalar field (DISF), a vector field (DICV), and a function of several variables (DFSV).

A function f of several variables (FVV), defined on an open subset Ω of $\mathbb{R}^n$ to $\mathbb{R}^m$ (with $n \geq 1$ and $m \geq 1$), is differentiable at an interior point a of Ω if there exists a linear transformation T_a: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ and a function E_a, defined as shown in Equation 1:

$$E_a: \Omega \subseteq R^n \rightarrow R^m, v \mapsto E_a(v) = E(a, v) \quad (1)$$

Such that,

$$f(a+v) = f(a) + T_a(v) + \|v\| E(a, v) \quad (2)$$

where $E_a \rightarrow 0$ as $v \rightarrow 0$. The transformation T_a is called the differential of f at a and is denoted by $Df(a)$ [1].

The above definition is equivalent to stating that the function f is differentiable at an interior point a of Ω if there exists a linear transformation T_a: $\mathbb{R}^n \rightarrow \mathbb{R}^m$ such that it satisfies the Equation 3 [21].

$$\lim_{x \to a} \frac{\|f(x) - f(a) - T_a(x-a)\|_m}{\|x-a\|_n} = 0 \quad (3)$$

The formulation and refinement of the GD of the concept of DFSV is as follows [22]:

4.1. Preliminary Constructions

A1. The concept of a FVV an object; upon decapsulation, the following processes are obtained, depending on the values of m and n in (4)

$$f: \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m, \qquad x \longmapsto y = f(x) \quad (4)$$

Case 1, real function of a real variable (RF), $n = 1$ and $m = 1$; case 2, vector function of a real variable (FVR) $n = 1$ and $m > 1$; case 3, real function of a vector variable or scalar field (SF), $n > 1$ and $m = 1$; case 4, v ector function of a vector variable or vector field (VF). $n > 1$ and $m > 1$as represented in expression (4).

A2. The understanding of linear transformation as an object, the limit of a function of a real variable as a process, the derivative of a function at a point as a process and the concept of the derivative function as an object.

4.2. Differential Scheme for a Real Function of a Real Variable (DIFR)

Given a function f, an RF defined on an open subset A of $\mathbb{R}$ and a point a in A:

B1. Graphical-Numerical. The action of geometrically interpreting the DIFR as the change in the height of the tangent line, $T = T(x)$, to the graph of $y = f(x)$ at the point $(a, f(a))$, when x varies in the interval $[a, a+h]$ with $h > 0$.

B2. Algebraic-Numerical. The action of calculating the differential of f at a. This involves: a) Decapsulating the object of the derivative of a function at a point; b) Verifying the existence of the linear map T_a, $T_a(\alpha u + \beta v) = \alpha T_a(u) + \beta T_a(v)$, for all $\alpha, \beta \in \mathbb{R}$, expressed in Equation 5:

$$T_a: A \subseteq \mathbb{R} \rightarrow \mathbb{R},\ u \mapsto T_a(u) = f'(a)(u) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot u \quad (5)$$

Furthermore, verify whether the error function $E_a(h)$, defined in (6), is continuous at 0 by checking that the $lim\, E_a(h) = 0$ as $h \to 0$.

$$E_a: A \subseteq \mathbb{R} \to \mathbb{R},\ h \mapsto E_a(h) = \begin{cases} \frac{f(a+h)-f(a)-Df(a)(h)}{|h|}, & si\ h \neq 0 \\ 0, & si\ h = 0. \end{cases} \quad (6)$$

B4. Algebraic-Analytical. To internalize the actions from B2 into a process that calculates the differential of the function f at the point a when an arbitrary point x is approximated to a.

B5. Graphical-Analytical. To coordinate the processes described in B3 and B4 into a new process that establishes that an RF f is differentiable at a point a as the possibility of:

Approximating $f(x)$ (for x sufficiently close to a) by the affine map: $x \mapsto f(a) + f'(a)(x - a)$, interpreting this approximation geometrically as the possibility of linearizing the function $y = f(x)$ at the point $(a, f(a))$, that is, approximating the graph of the function by its tangent line in neighborhoods of this point.

B6. Encapsulation. To encapsulate the process B5 into the mathematical object *real function of a real variable differentiable at a point* (DIFR), and to represent it as the linear map given by expression (7):

$$Df(a): \mathbb{R} \to \mathbb{R},\ h \mapsto Df(a)h = f'(a)h \quad (7)$$

4.3. Differential Scheme for a Vector Function of a Real Variable (DIFVR)

Given f, a FVR (vector function of a real variable) defined on an open subset A of $\mathbb{R}$ to $\mathbb{R}^2$, and an interior point a of A.

C1. Graphical-Numerical. The action of plotting the curve in the plane represents f, the point $(a, f(a))$, and the tangent vector to f at that point.

C2. Algebraic-Analytical. The action of calculating the increment of f, $\Delta f(a) = f(a+h) - f(a)$, as x varies in the interval $[a, a+h]$, and comparing it to the change produced by the tangent vector to f at $(a, f(a))$.

C3. Graphical-Analytical. To internalize the actions from C1 into a process that determines if f is differentiable and geometrically interprets that, as $x \to a$, the vector $f'(a)(x - a)$ approximates the increment vector $\Delta f(a) = f(a + h) - f(a)$, which tends to coincide with a segment of the curve.

C4. Algebraic-Analytical. To internalize the actions from C2 into a process to verify whether f is differentiable at a. This involves: a) Verifying that the operator $Df(a)$, given by (8), is linear; b)Verifying that the following $\lim_{|u|\to 0} \frac{\|f(a+u)-f(a)-Df(a)u\|_2}{|u|} = 0$, (where $||\cdot||_2$ denotes the Euclidean norm in $\mathbb{R}^2$ and $|\cdot|$ the absolute value in $\mathbb{R}$).

$$Df(a): \mathbb{R} \longrightarrow \mathbb{R}^2, \qquad x \mapsto Df(a)(x) = f'(a)(x) = (f_1'(a)x, f_2'(a)x) \quad (8)$$

C5. Algebraic-Analytical. To coordinate the processes from C3 and C4 into a process that, given a FVR $f: A \subseteq \mathbb{R} \to \mathbb{R}^m$, $m > 1$ and a point a in its domain, determines that f is differentiable at a if there exists a linear transformation $T_a: \mathbb{R} \to \mathbb{R}^m, m > 1$, such that: $T_a(u) = (f_1'(a), \ldots, f_m'(a)) \cdot u$. And that, $\lim_{|u|\to 0} \frac{\|f(a+u)-f(a)-Df(a)u\|_m}{|u|} = 0$.

Furthermore, this process geometrically interprets the differential as the possibility of linearizing the curve (existence of the tangent vector to the curve f at the point $f(a)$) in neighborhoods of $f(a)$.

C6. Encapsulation. To encapsulate the process C5 into the object *differential of a vector function of a real variable* (DIVR).

4.4. Differential Scheme for the Partial Derivative of a Scalar Field (SF)

Given f, an SF defined on an open subset A of $\mathbb{R}^2$ to $\mathbb{R}$, and a point (a, b) in A.

D1. Graphical-Analytical. The action of geometrically interpreting:

The partial derivative of f with respect to x as the slope of the tangent line to the graph of the function $g(x) = f(x, b)$ (which represents the trace of the surface $z = f(x, y)$ on the plane $y = b$) at the point $(a, b, f(a, b))$.

The partial derivative of f with respect to y as the slope of the tangent line to the graph of the function $h(y) = f(a, y)$ (which represents the trace of the surface $z = f(x, y)$ on the plane $x = a$) at the point $(a, b, f(a, b))$.

D2. Algebraic-Analytical. Let the unit vectors $\hat{e}_1 = (1, 0)$ and $\hat{e}_2 = (0, 1)$. The action of calculating:

The partial derivative of f with respect to x at the point (a, b), denoted as $\frac{\partial f}{\partial x}(a, b)$ or $f_x(a, b)$, according to expression (9).

The partial derivative of f with respect to y at the point (a, b), denoted as $\frac{\partial f}{\partial y}(a, b)$ or $f_y(a, b)$, according to expression (10).

$$\begin{aligned} g'(a) &= \lim_{t\to 0} \frac{g(a+t) - g(a)}{t} = \lim_{t\to 0} \frac{f(a+t, b) - f(a, b)}{t} \\ &= \lim_{t\to 0} \frac{f\big((a, b) + te_1\big) - f(a, b)}{t} = \frac{\partial f}{\partial x}(a, b) \end{aligned} \quad (9)$$

$$\begin{aligned} h'(b) &= \lim_{t\to 0} \frac{h(b+t) - h(b)}{t} = \lim_{t\to 0} \frac{f(a, b+t) - f(a, b)}{t} \\ &= \lim_{t\to 0} \frac{f\big((a, b) + te_2\big) - f(a, b)}{t} = \frac{\partial f}{\partial y}(a, b) \end{aligned} \quad (10)$$

D3. Graphical-Analytical Generalization. To repeat and internalize the actions from D1 into a process that, for different points $a = (a_1, \dots, a_n)$ and different SFs defined on open subsets of $\mathbb{R}^n$ with $n > 1$, assumes that the partial derivative of f with respect to each variable x_j $(j = 1, \dots, n)$ at the point a provides information about the behavior of f for values x near a, when x is constrained to the set $M = (a + t\hat{e}_j : t \in \mathbb{R})$ (the line passing through a in the direction of the unit vector $\hat{e}_j$.

D4. Algebraic-Analytical Generalization. To internalize the actions from D2 into a process that: given an open subset A of $\mathbb{R}^n$, a point $a = (a_1, \dots, a_n)$ in A, an SF $f: A \to \mathbb{R}$, and a direction vector $u = \hat{e}_j$ in $\mathbb{R}^n \setminus \{0\}$ (for $j = 1, \dots, n$), where $\{\hat{e}_1 = (1, 0, \dots, 0), \hat{e}_2 = (0, 1, \dots, 0), \dots, e_n = (0, 0, \dots, 1)\}$ is the standard basis of $\mathbb{R}^n$, calculates the partial derivative of f with respect to the variable x_j at the point a according to expression (11).

$$\frac{\partial f}{\partial x_j}(a) = \lim_{t \to 0} \frac{f(a + te_j) - f(a)}{t} \tag{11}$$

D5. Synthesis and Coordination. To coordinate the processes from D3 and D4 into a unified process that, given a real-valued function of a vector variable (SF), calculates and interprets (graphically, analytically, and numerically) the partial derivative of f with respect to each variable x_j $(j = 1, \dots, n)$ at the point a.

D6. Encapsulation and Representation. To encapsulate the process D5 into the mental object "*partial derivative (PD) of f with respect to the variable x_j at the point a*". This object is formally represented by the notation $\frac{\partial f}{\partial x_j}(a, b)$ and is interpreted in the following ways:

As the instantaneous rate of change of f with respect to the variable x_j, when the other variables x_k ($k \neq j$) are held constant.

Geometrically, as the rate of change of the function in neighborhoods of the point a in the direction of the unit vector $\hat{e}_j$ of the standard basis of $\mathbb{R}^n$. Algebraically, it is expressed by expression (12).

$$\frac{\partial f}{\partial x_j}(a) = \lim_{h \to 0} \frac{f(a_1, \dots, a_j + h, \dots a_n) - f(a_1, \dots, a_j, \dots, a_n)}{h} = \lim_{h \to 0} \frac{f(a + h\hat{e}_j) - f(a)}{h} \tag{12}$$

4.5. Differential Scheme for the Directional Derivative of a Scalar Field (DD).

Given a scalar field (SF) f, defined on an open subset A of $\mathbb{R}^2$ to $\mathbb{R}$, a point $P(x_0, y_0)$ in A, and a unit vector $u = (u_1, u_2)$.

E1. Graphical-Analytical. The action of geometrically interpreting the directional derivative of f at the point $P(x_0, y_0)$ in the direction of the unit vector u as the slope of the tangent line to the curve formed by the intersection of the surface $z = f(x, y)$ with the vertical plane that passes through P in the direction of u, evaluated at the point $(x_0, y_0, f(x_0, y_0))$.

E2. Algebraic-Analytical. The action of calculating the directional derivative of f at the point $P(x_0, y_0)$ in the direction of the unit vector u by applying expression (13).

$$D_u f(x_0, y_0) = \lim_{t \to 0} \frac{f((x_0, y_0) + t(u_1, u_2)) - f(x_0, y_0)}{t} \tag{13}$$

E3. Graphical-Analytical Generalization. To internalize the actions from E1 into a process that: given an open subset A of $\mathbb{R}^n$, a point $a = (a_1, \dots, a_n)$ in A, a scalar field (SF) f defined from A to $\mathbb{R}$, and a unit vector $u = (u_1, \dots, u_n)$ in $\mathbb{R}^n \setminus \{0\}$, represents and interprets the directional derivative of f at the point a in the direction of u as the instantaneous rate of change of f in the neighborhoods of the point a along the one-dimensional line $\{a + tu : t \in \mathbb{R}\}$.

E4. Algebraic-Analytical Generalization. To internalize the actions from E2 into a process to calculate the directional derivative of f at the point a in the direction of the unit vector u, verifying equation (14) for this purpose.

$$D_u f(a_1, \dots a_n) = \lim_{t \to 0} \frac{f(a_1 + tu_1, \dots, a_n + tu_n) - f(a_1, \dots, a_n)}{t} \tag{14}$$

E5. To coordinate the processes E3 and E4 into the process of, given a function f defined on an open set A in $\mathbb{R}^n$, a point $(a_1, \dots, a_n)$ in A, and a vector $u = (u_1, \dots, u_n)$ in $R^n \setminus \{0\}$, calculating the directional derivative of f at point a in the direction of vector u and providing a geometric interpretation of this derivative.

E6. To encapsulate the process E5 into the object *directional derivative of the function f at point a in the direction of the unit vector u*, and denote it as $\frac{\partial f}{\partial u}(a) = D_u f(a)$. This is interpreted as the rate of change of f at point a in the direction of the unit vector u and is calculated using ex pression (15).

$$\frac{\partial f}{\partial u}(a) = \lim_{h \to 0} \frac{f(a_1 + hu_1, \dots, a_n + hu_n) - f(a_1, \dots, a_n)}{h} = \lim_{h \to 0} \frac{f(a + hu) - f(a)}{h} \tag{15}$$

4.6. Scheme for the Derivative of a Real Function of a Vector Variable at a Point with Respect to a Vector, (DFVR).

F1. Unpack the directional derivative object of function f at point a in the direction of vector u into the process that, given f, a point a, and an arbitrary vector (not necessarily unit or from the canonical basis) w, finds if the limit given by expression (16) exists, and interprets $D_w f(a)$ as the rate of change of f at point a in the direction of an arbitrary vector w in $\mathbb{R}^n$.

$$D_w f(a) = \lim_{h \to 0} \frac{f(a_1 + hw_1, \dots, a_n + hw_n) - f(a_1, \dots, a_n)}{h} \tag{16}$$

F2. To encapsulate the process F1 into the object *derivative of the function f at point a with respect to the vector w*, by finding (if it exists) the limit given by expression (17).

$$D_w f(a) = \lim_{h \to 0} \frac{f(a + hw) - f(a)}{h} \quad (17)$$

4.7. Differential Scheme for a Scalar Field (DISF).

G1. Analytical-algebraic. Given a function of two variables, perform the action of verifying that the existence of the function's derivative at a point with respect to any vector does not imply that the function is continuous at that point. For example, for the function $f: \mathbb{R}^2 \to \mathbb{R}$ defined by: $f(x,y) = \frac{xy^2}{x^2+y^4}$ if $x \neq 0$, $f(0,y) = 0$, directional derivatives exist at (0,0) for any direction vector; however, the function is not continuous at (0,0).

G2. Analytical-algebraic. Given a function of two variables, perform the action of verifying that the existence of the derivative of f at a point with respect to any vector w, combined with the continuity of the function at that point, does not imply that the directional derivative (as a function of the vector w) is a linear operator. For example, for the function $g: \mathbb{R}^2 \to \mathbb{R}$ defined by: $g(x,y) = \frac{x^2 y}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and 0 if $(x,y) = (0,0)$, the directional derivatives exist at (0,0) for any direction vector and the function is continuous at (0,0); however, the directional derivatives at (0,0) is not linear with respect to direction vector.

G3. Analytical-algebraic. Given the function of two variables whose directional derivative at a point is a linear function of the direction vector and which is discontinuous at that point, perform the action of verifying that the existence of a linear directional derivative at a point does not imply the continuity of the function at that point. For example, for the function $h: \mathbb{R}^2 \to \mathbb{R}$ defined by: $h(x,y) = \frac{x^3 y}{x^6+y^2}$ if $(x,y) \neq (0,0)$, 0 if $(x,y) = (0,0)$, the directional derivative at (0,0) exists and is a linear function of the direction vector; however, the function is discontinuous at the origin.

G4. Repeat actions G1, G2, and G3 with various function examples, and internalize the process that, given a scalar field f and a point a in its domain, the existence of the directional derivative of f at a (even for every direction) does not guarantee either the continuity and differentiability of f at a, as the case for real-valued functions of a single real variable.

G5. Encapsulate the process G4 into the property: For scalar fields, the existence of directional derivatives does not imply continuity or differentiability.

G6. Graphical-analytical. Given a scalar field f defined from $A \subseteq \mathbb{R}^2$ to $\mathbb{R}$ and a point (a,b), perform the action of interpreting the differential of f at (a,b) geometrically as the change in height of the tangent plane to the surface $z = f(x,y)$ at the point $(a,b,f(a,b))$ when moving from the point (a,b) to the point $(a + \Delta x, b + \Delta y)$.

G7. Analytical-algebraic. Given a scalar field f defined from $A \subseteq \mathbb{R}^2$ to $\mathbb{R}$ and a point (a,b), perform the action of calculating the differential of f at (a,b). This is achieved by verifying whether the limit $(\Delta x, \Delta y) \to (0,0)$ of $E(\Delta x, \Delta y)$ is 0 (where E is the error of the linear approximation, equivalent to verifying Equation 18. If this holds, then the differential of f at (a,b) is given by the dot product of the gradient and the increment vector:

$Df(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)$.

$$\lim_{\|(\Delta x, \Delta y)\| \to 0} \frac{f(a + \Delta x, b + \Delta y) - f(a,b) - \frac{\partial f}{\partial x}(a,b)\Delta x - \frac{\partial f}{\partial y}(a,b)\Delta y}{\|(\Delta x, \Delta y)\|} = 0 \quad (18)$$

G8. To internalize action G6 into the process of geometrically interpreting the differential of f at a, denoted $Df(a)$, defined as the linear map from $\mathbb{R}^n$ to $\mathbb{R}$ that is used to linearize the function f around point a.

G9. To internalize action G7 into the process that, given an open set $A \subseteq \mathbb{R}^n$, a function $f: A \to \mathbb{R}$ and an interior point a of A, determines whether f is differentiable at a. If it is, it calculates its differential using expression (19). The vector that defines this linear map (i.e., the vector whose components are the partial derivatives) is called the gradient vector of the function f at point a, denoted $\nabla f(a)$.

$$Df(a) = \nabla f(a) = \left(\frac{\partial f(a)}{\partial x_1}, \cdots, \frac{\partial f(a)}{\partial x_n}\right) \quad (19)$$

G10. To coordinate processes G8 and G9 into the process that, given an open set $A \subseteq \mathbb{R}^n$, a function $f: A \to \mathbb{R}$, and an interior point a of A, determines whether f is differentiable at a and, if it is, calculates its differential. This process provides both a geometric and an analytical interpretation of the differential, establishing the relationships between the fundamental concepts that configure the DISF scheme. Furthermore, it identifies and interrelates the following elements: the scalar field (SF) f, the linear map $T_a: \mathbb{R}^n \to \mathbb{R}$ (from the definition of differentiability), and the gradient vector $\nabla f(a)$, which are related through the function defined in expression (20).

$$E_a: A \subseteq \mathbb{R}^n \to \mathbb{R}, v \mapsto E_a(v) = E(a,v) = \begin{cases} \frac{f(a+v) - f(a) - T_a(v)}{\|v\|_n}, & v \neq 0 \\ 0, & v = 0 \end{cases} \quad (20)$$

The afore mentioned relationships yield the one described by Equation 21. This allows us to conclude that if there exists a linear transformation T_a and a function $E(a,v)$ such that for all vectors v with $||v|| < r$ (where $r > 0$), it holds that $E(a,v) \to 0$, as $v \to 0$ then the scalar field f is differentiable at point a.

$$f(a+v) = f(a) + T_a(v) + \|v\| E(a,v) \quad (21)$$

Furthermore, the afore mentioned process can be synthesized equivalently by verifying whether the conditions described by (22) are satisfied; if so, then the scalar field f is differentiable at a.

$$\lim_{\|v\|\to 0} \frac{f(a+v) - f(a) - T_a(v)}{\|v\|} = 0 \tag{22}$$

G11. To encapsulate the processes from G10 into the object of a *differentiable scalar field f at point a* (DISF).

G12. To unpack the DISF concept to perform the process that, given an open set $A \subseteq \mathbb{R}^n$, a function $f: A \to \mathbb{R}$, and an interior point a of A, proves that if f is differentiable at a, then f is continuous at a.

G13. To encapsulate the process G12 into the mathematical object: *for scalar fields, differentiability implies continuity* (TDICO).

G14. To unpack the DISF concept to perform the process that, given a subset $A \subseteq \mathbb{R}^n$, a function $f: A \to \mathbb{R}$, and an interior point a of A, proves that if f is differentiable at a, then the directional derivative of f at a exists for any direction vector w, denoted $D_w f(a)$.

G15. To encapsulate the process G14 into the mathematical object: *for scalar fields, differentiability implies the existence of directional derivatives* (TDIDE).

G16. To unpack the DISF concept to perform the process of finding the best linear estimation of a scalar field f, defined in tabular form, at points not recorded in the table.

4.8. Differential Scheme for a Vector Field (DICV).

Given an open subset A of $\mathbb{R}^2$, a vector field $f: A \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, and a point a in A:

H1. (Analytical-algebraic). Perform the action of determining whether f is differentiable at a by checking if each of its scalar component fields (f_1, f_2, f_3), defined from $\mathbb{R}^2$ to $\mathbb{R}$, are differentiable at a.

H2. (Analytical-algebraic). Perform the action of establishing whether f is differentiable at a by verifying if the linear map $T_a: \mathbb{R}^2 \to \mathbb{R}^3$ (defined as the candidate for the differential) satisfies the linearity condition: for scalars α, β and vectors u, v in $\mathbb{R}^2$, it holds that: $T_a(\alpha u + \beta v) = \alpha T_a(u) + \beta T_a(v)$.

$$\mathrm{T_a}: \mathbb{R}^2 \to \mathbb{R}^3$$

$$u \to T_a(u) = Df(a)(u) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) \\ \frac{\partial f_3}{\partial x_1}(a) & \frac{\partial f_3}{\partial x_2}(a) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{23}$$

Furthermore, verify whether the function defined in (24), $E(a, v)$, tends to 0 as the norm of vector v tends to 0 (i.e., as $||v||_2 \to 0$).

$$E_a: \mathrm{A} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$v \longmapsto E_a(v) = E(a, v) = \begin{cases} \frac{\|f(a+v) - f(a) - Df(a)(v)\|_3}{\|v\|_2}, & v \neq 0 \\ 0, & v = 0 \end{cases} \tag{24}$$

Let A be an open set in $\mathbb{R}^n$, $n > 1$, a vector function $f: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, and a point a in A.

H3. To internalize the actions from H1 into the process of establishing whether the vector field f is differentiable at a by determining if each of its scalar component fields $(f_1, \ldots, f_m)$, defined from $\mathbb{R}^n$ to $\mathbb{R}$, are differentiable at a.

H4. Analytical-algebraic. To internalize the actions from H2 into the process that determines whether f is differentiable at a by verifying if the linear map T_a defined in (25) is linear; that is, for all scalars α, β and all vectors u, v in $\mathbb{R}^n$, it holds that:

$T_a(\alpha u + \beta v) = \alpha T_a(u) + \beta T_a(v)$.

$$\mathrm{T_a}: \mathbb{R}^n \to \mathbb{R}^m$$

$$u \to T_a(u) = Df(a)(u) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(a) & \cdots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \tag{25}$$

Furthermore, verify whether the vector-valued function defined in (26), $E(a, v)$, satisfies the following condition: $\lim_{\|v\|_n \to 0} \frac{E\|a,v\|_m}{\|v\|_n} = 0$.

$$E_a: \mathrm{A} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$v \longmapsto E_a(v) = E(a, v) = \begin{cases} \frac{\|f(a+v) - f(a) - Df(a)(v)\|_m}{\|v\|_n}, & v \neq 0 \\ 0, & v = 0 \end{cases} \tag{26}$$

H5. To encapsulate the process from H4 into the object *differential of a vector field f at a point a* (DICV), which is verified by confirming that the fundamental condition expressed in (27) is satisfied.

$$\lim_{\|v\|_n \to 0} \frac{\|f(a+v) - f(a) - T_a(v)\|_m}{\|v\|_n} = 0 \tag{27}$$

Below are characterized the levels and sublevels of the development of the scheme of the Derivative of Functions of Several Variables (DFSV)

Intra Level. Students focus their actions on repetitive actions or operations between internal elements of the DIFR (differential of a real function) without relating it to other objects of the scheme.

Intra Sublevel 1. i) (A1, A2 of the GD): Do not use the mathematical elements DIFR, DIFVR, DISF, and DICV; *ii*) Recall internal elements that constitute DIFR, DIFVR, DISF, or DICV, some with errors and in a single mode of representation (graphical or algebraic), but without establishing any relationships between them.

Intra Sublevel. *iii*) Recall some elements of DIFR in isolation, such as the real function (RF), the tangent to a function at a point, the increment of the independent variable h, the increment of the function f at point a ($\Delta f(a) = f(a+h) - f(a)$), in a single mode of representation; *iv*) (B1, B2 of the GD): Perform actions without internalizing them as processes, with the internal elements of DIFR; for example, finding the derivative of a function by applying differentiation rules; *v*) (B3, B4, B5 of the GD): Represent the construction of the derivative of an RF in algebraic and graphical form, managing to internalize some actions.

Inter Level. Students at this level can relate the differential of a real function (DIFR) with contiguous elements, such as vector functions of a real variable (FVR) and real functions of a vector variable (RF). These relationships arise when they recognize that differentiation is generalized from RFs to vector functions of a real variable, as the DIFR of each of their components.

Furthermore, they understand that for real functions of a vector variable or scalar fields (SF), the objects directional derivative (DD) and partial derivative (PD) are not a satisfactory generalization of DIFR, because for these types of functions, differentiability does not imply continuity.

The following sublevels of the Inter level were established: Inter 1, Inter 2, and Inter 3.

Inter Sublevel 1. *vi*) (B6, C1, C2 of the GD): Recognize DIFVR as a mathematical element contiguous to DIFR. This is evidenced when the student has understood the vector function of a real variable (FVR), defined from an open set $J \subseteq \mathbb{R}$ to $\mathbb{R}^{\mathrm{m}}$, as an object, and decomposes it into m RFs, calculating the DIFR for each of them to construct the contiguous element: the differential of a vector function of a real variable (DIFVR), composed of the m DIFRs applied to the same real number; *vii*) Establish relationships between the contiguous elements DIFR and DIFVR. This occurs when the student has understood the elements RF, FVR, DIFR, and DIFVR as processes or objects, and relates them through the logical condition: if the DIFRs that make up the DIFVR are not all zero, then they perform actions that transform these elements to represent algebraically the tangent space to the FVR defined by parametric equations (corresponds to C3, C4, C5 of the GD).

Inter Sublevel 2. *viii*) Identify the mathematical elements DD and PD as similar in nature to the derivative of a real function (DIFR) (D1, D2, D3, D4, E1, E2, E3, E4 of the GD). This is evidenced when the student extends the concept of the derivative of an RF to scalar and vector fields to calculate the DD or PD in different representational forms; *ix*) Generalize the differentiation processes of an RF to an SF or a VF (D5, D6, E5, E6, F1, F2 of the GD). This is manifested when they relate the elements DIFR, DD, PD and manage to generalize them in the process that calculates the derivative of a function of several variables at a point with respect to a direction vector, understanding this derivative as a transformation that, when linear, is calculated as the dot product of the gradient vector evaluated at the point with a direction vector in $\mathbb{R}^{\mathrm{n}}$, using this concept to solve problem situations.

Inter Sublevel 3. *x*) Recognize properties of continuity, differentiability, and differentiability of the mathematical elements RF and SF (G1, G2, G3, G4, G5 of the GD). This is externalized when reflecting on the relationship "differentiability implies continuity", which holds for RFs but not for scalar and vector fields, thus requiring the extension of the DIFR concept to DICE to guarantee the continuity of the SF; *xi*) Recognize when an SF is differentiable (G6, G7, G8, G9, G10 of the GD).

Trans Level. This level includes students who manage to generalize the differentiation process to functions of several variables and establish syntheses in the scheme when they relate mathematical elements to make inferences in solving problem situations. The sublevels Trans 1 and Trans 2 are distinguished.

Trans Sublevel 1. *xii*) Establish relationships between the mathematical element DICE, understood as a process or object, with DICV (B6, G11, H1, H2, H3, H4 of the GD). This is manifested when they have understood the element DISF as an object and unpack it into the process that verifies whether a VF is differentiable at a point; *xiii*) Present initial attempts at synthesis between the mathematical elements DIFR, DIFVR, DISF (H5 of the GD). This is evidenced when attempting to generalize the differentiation processes of an RF, FVR, and SF to calculate the differential of a vector field.

Trans Sublevel 2. *xiv*) Establish logical relationships between the elements that configure the DFSV scheme to infer properties (B6, G4, G5, G12, G13, G14, G15, G16 of the GD); *xv*) Establish coherence of the DFSV scheme (B6, C6, D6, G11, H5 of the GD). This is manifested when generalizing and synthesizing the properties satisfied by the mathematical elements to determine when an FVV is differentiable at a point and establish its differential according to the dimensions of the spaces between which the function is defined (DIFR, DIFVR, DISF, and DICV).

The transition from the Intra to the Inter level occurs when the student unpacks the DIFR object into processes and performs synthesis, generating new information that allows them to calculate derivatives of vector functions of a real variable and real functions of a vector variable.

The transition from Inter to Trans is perceived when the student unpacks DISF and applies processes to calculate the derivative of a vector field and the derivative of a function of several variables.

In this regard, Table 1 presents a summary of characteristics *i* to *xv*, corresponding to each level and sublevel of the development of the DFSV scheme, and the placement of students with pseudonyms E1 to E9, according to the performances marked with an X, from which the following can be inferred.

Students are placed at different development levels depending on the number of mathematical elements they recalled, the relationships they established between them, and the forms of representation, when they attempted to solve the designed tasks that required a certain demand of information for their understanding, in terms of the mental structures and mechanisms described by the GD.

Students E5, E6, E7, and E8 recall mathematical elements in an isolated manner because they present characteristics discontinuously across different development levels. The difference in the development levels of the scheme arises because some only recall the mathematical elements, while others, in addition to recalling them, relate and apply them. Within the same level, differences exist due to the number of elements used correctly and the logical relationships established between them to solve the tasks.

Students who find it easy to recognize mathematical elements in different representational forms find it easier to establish logical relationships and solve problem situations. However, a tendency to use algebraic representation, formulas, and expressions is noted, but when functions are represented in numerical or tabular form, they have difficulty interpreting the concepts and solving problems.

Table 1.
Student placement in the levels of the DFVV scheme and their characteristics.

Niveles	**Intra**					**Inter**						**Trans**			
Subnivel	**Intra 1**		**Intra**			**Inter 1**		**Inter 2**		**Inter**		**Trans 1**		**Trans**	
Estu.\Caracate.	*i*	*ii*	*iii*	*iv*	*v*	*vi*	*vii*	*viii*	*ix*	*x*	*xi*	*xii*	*xiii*	*xiv*	*xv*
E1			X	X	X	X	X	X	X	X	X	X	X	X	X
E2			X	X	X	X	X	X	X	X	X	X	X	X	
E3			X	X	X	X	X	X	X	X	X	X	X		
E4			X	X	X	X	X	X	X	X	X	X			
E5			X		X	X	X	X				X			
E6			X					X	X						
E7			X	X	X	X		X							
E8			X	X	X										
E9		X													

5. Discussion

Some difficulties in understanding the DFSV arise when students must relate the mathematical elements that constitute the concept and perform translations between different representation registers and the operations within them to solve exercises and problem situations.

This is due to the cognitive complexity of the multiple interactions between historical, theoretical, epistemological, pedagogical, and didactic aspects, which are prerequisites and form the schema of the DFSV. This complexity is well-documented in prior research concerning fundamental concepts such as: Infinitesimal [23, 24] function [25-27] limit [28-32] derivative [32-35] diferentiability [36-39] and integral [40-44].

To characterize and analyze the development of the differential schema, a methodology was adopted that articulated an initial genetic decomposition (GD) with the study of student performance. The process involved the implementation of an ACE, using questionnaires and interviews as data collection instruments, in accordance with the model presented in Boigues, et al. [40].

The analysis of the understanding of the differential of a function of several variables (DFVV) reveals that most students are at an intra-level of scheme development. Their performance is characterized by a mechanical application of algorithms for calculating derivatives, gradients, and Jacobian matrices, utilizing action structures typical of this level. This situation highlights the inherent duality in its teaching: the dilemma between emphasizing mere procedural execution or fostering a more rigorous conceptual analysis that facilitates progression to the inter and trans levels, an objective supported by the research [44, 45]**.**

Regarding the schema of the differential for scalar fields, it is observed that students understand it as a reconstructive generalization of the derivative of a single-variable function. In this extended schema, the property of differentiability at a point automatically implies continuity at that point—a relationship that aligns with the framework for the generalization of mathematical concepts described by Harel and Tall [46].

6. Conclusions

The GD of the concept under study allowed for the design of a pathway to understanding the concept of the derivative of functions of several variables (DFSV) at an interior point of the function's domain. This concept signifies the possibility of approximating the function in neighborhoods of a point by a linear transformation.

This procedure requires, starting from the object $f'(a)$ (derivative in one variable), unpacking it to define the linear transformation $Df(a)$ (differential) and verifying that it approximates the function at points near a, which is proven by analyzing that the approximation error tends to zero.

The analysis of the understanding of the derivative of functions of several variables (DFVV) is complex, as it requires the articulation of several prior schemas such as: topology in $\mathbb{R}^n$ (with concepts of open set, interior point, and neighborhood); the concept of a function (which allows for analyzing the type of function based on its domain and codomain); the limit and its generalization to functions of several variables; the derivative of a real function of a real variable; and linear transformations, defined by the derivative operator, on vector spaces (concepts of norm, unit vectors, and inner product).

The interaction of these schemas is essential to define and analyze the error function in several variables. The ultimate purpose is to establish whether a function is differentiable at a point and, if so, to determine its differential.

Furthermore, the mental constructions of APOS theory (Action, Process, Object, and Schema) for differentiability can be characterized by using a variety of functions, both explicitly defined and generalized. These include constant, linear, polynomial, rational, and composite functions.

It is crucial to use functions that exhibit issues of continuity or differentiability at some points in their domain. This analysis is fundamental to enabling students to complete the transition from a procedural understanding to the conceptualization of a mathematical object. This leap will allow them to meaningfully relate the concepts of continuity, derivability, differentiability, and the inherent properties of differentiable functions.

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